ISRAEL JOURNAL OF MATHEMATICS, Vol. 64, No. 2, 1988

THE LARGEST PROJECTIONS OF REGULAR POLYTOPES

BY
P. FILLIMAN
Department of Mathematics, Western Washington University, Bellingham, WA 98225, USA

ABSTRACT
The projections of the regular n-dimensional simplex and crosspolytope into R^k with the largest k-volume are determined here for the cases $k = 2$, $n \geqq 2$ and $k = 3$, $4 \leqq n \leqq 6$. The proofs involve a combination of exterior algebra and computer gradient methods.

Introduction

This paper applies the methods of [8] to investigate the projections of the regular n-dimensional simplex and crosspolytope into R^k with the greatest k-volume.

Section 2 contains the proof that the largest planar projections are squares. Sections 3 and 4 contain the results for $k = 3$ and $n \leqq 6$, which generally involve splitting up the problem among combinatorial types. These results suggest that the maximal projections stabilize after some n, and Section 5 gives upper bounds which support this conjecture. The final section contains tables listing the known solutions.

1. Definitions

Let $E = \{e_1, \dots, e_n\}$ be the standard orthonormal basis of R^n, real n-dimensional Euclidean space. The regular simplex, crosspolytope and cube can be written conveniently as

$$T^{n-1} = \operatorname{conv} E, \tag{1.1}$$

Received December 22, 1987 and in revised form September 7, 1988

$$X^n = \operatorname{conv}\{-E, E\} \tag{1.2}$$

and

$$C^n = \sum_{i=1}^{n} [e_i], \tag{1.3}$$

where the sum is the Minkowski sum of the line segments

$$[e_i] = \operatorname{conv}\{e_i/2, e_i/2\}, \qquad i = 1, \dots, n.$$

Let R_n be the regular n-gon with circumradius 1 centered at the origin. If we take R_n to lie in the plane lin $\{e_1, e_2\} \subset R^3$, then

$$D_n = \operatorname{conv}\left\{ \pm \frac{1}{\sqrt{2}} e_3, \sqrt{\frac{2}{n}} R_n \right\} \tag{1.4}$$

and

$$D'_n = \operatorname{conv}\left\{ \pm e_3, \sqrt{\frac{2}{n}} R_{2n} \right\} \tag{1.5}$$

are symmetrical bipyramids. We shall see later that these are in many cases maximal projections of T^{n+1} and X^{n+1} respectively. The volumes of these figures are

$$V(R_n) = \frac{n}{2} \sin\left(\frac{2\pi}{n}\right), \tag{1.6}$$

$$V(D_n) = \frac{\sqrt{2}}{3} \sin\left(\frac{2\pi}{n}\right) \tag{1.7}$$

and

$$V(D'_n) = \frac{4}{3} \sin\left(\frac{\pi}{n}\right). \tag{1.8}$$

In [8], the problem of determining the largest projections Max $\Pi^k(Q)$ of an n-dimensional polytope Q into R^k is related to the geometry of the Grassmannian $G(k, n)$ of oriented k-dimensional subspaces of R^n. The remainder of this section is a summary of [8].

The exterior algebra $\Lambda_k R^n$ is a real vector space of dimension $\binom{n}{k}$. The Grassmannian $G(k, n)$ can be embedded as a submanifold of the unit sphere in $\Lambda_k R^n$ by sending a k-plane $L = \operatorname{lin}\{x_1, \dots, x_k\}$ to $\xi = x_1 \wedge \cdots \wedge x_k$, where $\{x_1, \dots, x_k\}$ is any oriented orthonormal basis of L. This identification allows us to transform functions on k-planes to functions on the corresponding unit k-vectors in $\Lambda_k R^n$.

Let Q be a convex polytope in R^n and let $P = \Pi(Q:L)$ be the orthogonal projection of Q into a k-plane L. Denote by $V(Q:L) = V(P)$ the k-dimensional volume of P. The key to understanding $V(Q:L)$ is (see [8, Section 2])

THEOREM 1. *The function $V(Q:L)$ is piecewise linear on $G(k, n)$.*

The pieces correspond to combinatorial types. One piece consists of the k-planes in which projections of Q have a fixed combinatorial type, and is denoted by $\chi(\Phi)$. For each region there is a "projection form" $\Phi \in \Lambda_k R^n$ such that

$$V(Q:L) = \langle \Phi, \xi \rangle \tag{1.9}$$

holds locally, that is, when $\xi \in \chi(\Phi)$. The right-hand side is the usual inner product in $\Lambda_k R^n$.

The "comass" of an arbitrary k-vector $\Phi \in \Lambda_k R^n$ is

$$\| \Phi \| = \{\max \langle \xi, \Phi \rangle \mid \xi \in G(k, n)\}. \tag{1.10}$$

The set

$$G(\Phi) = \{ \xi \in G(k, n) \mid \langle \Phi, \xi \rangle = \| \Phi \| \} \tag{1.11}$$

is the face of $G(k, n)$ with outer normal Φ. Thus $G(\Phi)$ equals the vectors in $G(k, n)$ which maximize $\langle \xi, \Phi \rangle$. However if Φ is a projection form, formula (1.9) implies $\langle \xi, \Phi \rangle = V(Q:L \rangle$ for $\xi \in \chi(\Phi)$. Combining these remarks gives

PROPOSITION 2. *If $G(\Phi) \subset \chi(\Phi)$, then the k-vectors in $G(\Phi)$ correspond to k-planes in R^n containing the projections which maximize volume locally, i.e. among all projections with the given combinatorial type. If $G(\Phi) \not\subset \chi(\Phi)$, then $\| \Phi \|$ remains an upper bound on the volumes of projections of the given type.*

Proposition 2 will be used in Sections 3 and 4 to find Max $\Pi^k(Q)$ as follows: For each combinatorial type of projection, we determine Φ, $\| \Phi \|$ and $G(\Phi)$, and decide whether $G(\Phi) \subset \chi(\Phi)$. Then the solution is found by taking the largest comass among those types for which $G(\Phi) \subset \chi(\Phi)$.

Suppose, for example, that Q is the regular simplex T^n. Since every combinatorial type of polytope with n or fewer vertices appears as a projection of T^n, we could follow the procedure above for all these combinatorial types. However, Theorem 10 in [8] implies that the maximal projections are simplicial, so one only needs to check the simplicial types. A similar process works for X^n, except that the projections are always centrally symmetric polytopes with $2n$ or fewer vertices. A description of the method used to obtain the projection form and comass of a combinatorial type will be postponed until Section 4.

2. Projections into R^2

Instead of working directly with the projections of T^n, we use the fact that $\operatorname{Max} \Pi^k(T^n)$ is the collection of polytopes $P = \operatorname{conv}\{x_0, \dots, x_n\}$ which have the largest volume under the condition

$$\sum_{i=0}^{n} |x_i|^2 = k. \tag{2.1}$$

(see [8, Theorem 6]). A similar result holds for X^n and C^n. According to [8, Proposition 2], a maximal projection must lie in a subspace parallel to aff T^n in R^{n+1}. Hence, $\Sigma x_i = 0$ and $0 \in \operatorname{int} P$. Also, the number of vertices of P must lie between $k+1$ and $n+1$.

Now, suppose $k = 2$ and P has m vertices which we label in counterclockwise order about the origin. Then

$$V(P) = \frac{1}{2} \sum_{i=0}^{m-1} |x_i|\,|x_{i+1}| \sin \theta_i, \tag{2.2}$$

where θ_i is the angle between x_i and x_{i+1} ($x_m = x_0$). Let

$$\alpha_i = \frac{|x_i|\,|x_{i+1}|}{\alpha}, \qquad \text{where } \alpha = \Sigma |x_i|\,|x_{i+1}|.$$

The Cauchy–Schwarz inequality implies

$$\alpha^2 \leqq (\Sigma |x_i^2|)^2 = 4 \quad \text{and} \quad \alpha \leqq 2, \tag{2.3}$$

with equality if and only if $|x_i| = |x_j|$, $0 \leqq i, j \leqq m$. Substituting (2.3) in (2.2) gives

$$V(P) = \frac{\alpha}{2} \Sigma \alpha_i \sin \theta_i \leqq \Sigma \alpha_i \sin \theta_i.$$

Since $\Sigma \alpha_i = 1$ and $\sin \theta$ is concave on $[0, \pi]$, Jensen's inequality implies

$$\Sigma \alpha_i \sin \theta_i \leqq \sin(\Sigma \alpha_i \theta_i).$$

Therefore

$$V(P) \leqq \sin(\Sigma\, \alpha_i \theta_i) \leqq 1, \tag{2.4}$$

with equality if and only if $m = 4$, $\theta_i = \pi/2$ and $|x_i| = 1/\sqrt{2}$. In this case P is a unit square.

A projection of X^n into the plane is a centrally symmetric polygon P with at most $2n$ vertices. An argument like the one above shows that

$$V(P) \leqq 2, \tag{2.5}$$

with equality if and only if $P = X^2$.

3. Projections of T^n into R^3

Martini and Weissbach [9, p. 165] show that the largest 3-dimensional projection of T^4 is obtained by projecting it in the direction of the sum of any two or three of the normals to its facets. Although they give the volume of the projection, they do not describe it explicitly. In doing so, we show how to determine a projection of T^n given coordinates for the plane containing the projection.

A normal can be calculated by subtracting the centroid of T^4 from a vertex, e.g.

$$u_1 = \tfrac{1}{5}(1, 1, 1, 1, 1) - (1, 0, 0, 0, 0) = \tfrac{1}{5}(-4, 1, 1, 1, 1).$$

Adding $u_2 = \frac{1}{5}(1, -4, 1, 1, 1)$ gives

$$u = u_1 + u_2 = \tfrac{1}{5}(-3, -3, 2, 2, 2). \tag{3.1}$$

The projection lies in the 3-plane L in R^5 orthogonal to $\mathrm{lin}\{u, (1, 1, 1, 1, 1)\}$. An orthonormal basis for L consists of the rows of the matrix

$$\mathbf{X} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{6} & 1/\sqrt{6} & \sqrt{2/3} \end{bmatrix}. \tag{3.2}$$

Fortunately, a set of coordinates for the vertices of $\Pi(T^4 : L)$ is given by the columns of $\mathbf{X}$ (see [8, 4.1]). This figure is D_3, a bipyramid based on a regular triangle, and its volume is $1/\sqrt{6}$.

A maximal projection of T^5 into R^3 is a simplicial polytope P with at most 6 vertices and 8 faces. If Φ is its projection form, then the Cauchy–Schwarz inequality implies

$$V(P) = \langle \xi, \Phi \rangle \leqq |\xi||\Phi| = |\Phi|. \tag{3.3}$$

In the next section, we shall illustrate how to determine the projection form Φ of a given combinatorial type. It turns out that Φ has one nonzero component (of magnitude 1/3!) for each face of P. Hence

$$V(P) \leqq |\Phi| \leqq \sqrt{8/3!} = \sqrt{2/3}, \tag{3.4}$$

with equality if and only if $\xi = c\Phi$, $c \in R^+$.

The k-vector ξ has the special property that it can be written as the wedge product of some three vectors in R^5, in particular, a set which spans the 3-plane containing P. Any such k-vector is said to be "decomposable" (or "simple"). Thus if equality were to hold in (3.3), then Φ would be a multiple of ξ and so Φ would also be decomposable. Of the two simplicial combinatorial types with 6 vertices, only the form associated with the regular octahedron D_4 is simple. Therefore, equality holds in (3.4) only when $P = D_4$.

The largest projection of T^6 into R^3 is also D_4, since the comass of any type with 7 vertices is smaller than $\sqrt{2/3}$ (see Table 3). Among projections with 7 vertices, the polytope D_5 has the greatest volume (see Figure 3.5). Type 3.1 has the interesting property that its face $G(\Phi)$ consists of two points, only one of which corresponds to a polytope with this combinatorial type.

Remarkably, the figures D_3, D_4 and D_5 are similar to the solutions in [1] to the problem of maximizing the convex hull of 5, 6, or 7 points on the unit 3-sphere.

4. Projections of X^n into R^3

A projection of X^n into R^3 is a centrally symmetric polytope with $2n$ or less vertices. Tables 4, 5 and 6 give the comass of every type of simplicial projection except for the icosahedron when $n \leqq 6$. Since the comass is in every case smaller than $4/3 = V(X^3)$, Max $\Pi^3(X^n) = X^3$ for $4 \leqq n \leqq 6$. In this section we give some samples of the calculation of comass in these tables.

Let $e_1, \dots, e_n$ be the standard orthonormal basis of R^n. Then the simple k-vectors

$$\begin{gathered} e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_k}, \qquad \lambda \in \Lambda(n,k) \text{ where} \\ \Lambda(n,k) = \{(\lambda_1, \dots, \lambda_k) \mid 1 \leqq \lambda_1 < \cdots < \lambda_k \leqq n\} \end{gathered} \tag{4.1}$$

form an orthonormal basis of $\Lambda_k R^n$. We can write a k-vector $\Phi \in \Lambda_k R^n$ as

(4.2) $$\Phi = \sum_{\Lambda(n,k)} \Phi_\lambda e_\lambda, \qquad \Phi_\lambda \in R.$$

We now describe how to find the projection form Φ of the combinatorial type represented by Fig. 4.1. This figure is half the Schlegel diagram of a (centrally symmetric) projection of X^4 into some 3-plane L. The vertex labeled i shall be the projection of e_i into L, denoted by $x_i = \Pi(e_i : L)$. Each face $\operatorname{conv}\{x_{\lambda_1}, x_{\lambda_2}, x_{\lambda_3}\}$ contributes a factor of $\pm 1/3!$ to Φ_λ, with the sign depending upon the orientation of the basis $\{x_{\lambda_1}, x_{\lambda_2}, x_{\lambda_3}\}$ of R^3. Also, since e.g.

$$e_1 \wedge e_2 \wedge (-e_3) = -e_{123},$$

each of the opposing vertices $-x_1$, $-x_2$ and $-x_3$ contributes a factor of -1. For example, in computing Φ_{123} we consider the four faces

$$\operatorname{conv}\{x_1, x_2, x_3\}, \quad \operatorname{conv}\{-x_1, x_2, -x_3\},$$

$$\operatorname{conv}\{x_1, x_2, -x_3\} \quad \text{and} \quad \operatorname{conv}\{-x_1, x_2, x_3\}.$$

The first two have positive orientations and the last two have negative orientations, so

$$\Phi_{123} = 4 \cdot 1/3! = 2/3.$$

Altogether

(4.3) $$\Phi = 2/3(e_{123} + e_{134} + e_{124}).$$

Since $G(3, 4)$ is isomorphic to the unit 3-sphere in $\Lambda_3 R^4$, the comass of Φ (see 1.10) equals its Euclidean norm $|\Phi| = 2/\sqrt{3}$. The corresponding face $G(\Phi)$ is the unit 3-vector

(4.4) $$\xi = 1/\sqrt{3}(e_{123} + e_{124} + e_{134}).$$

It remains only to find the 3-plane L corresponding to ξ, and the projection $P = \Pi(X^4 : L)$. The standard procedure for determining L is to consider the matrix

(4.5) $$\mathbf{X}' = \begin{bmatrix} 1 & 0 & 0 & x_{14} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \end{bmatrix}.$$

The wedge product of the rows of $\mathbf{X}'$ is

(4.6) $$e_{123} + x_{34} e_{124} - x_{24} e_{134} + x_{14} e_{234}.$$

Comparing (4.4) and (4.6), we find that $x_{14}=0$, $x_{24}=-1$ and $x_{34}=1$. Substituting these values in (4.5) and then using the Gram–Schmidt process on the rows of $\mathbf{X}'$, we obtain the matrix

$$\mathbf{X}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{6} & \sqrt{2/3} & 1/\sqrt{6} \end{bmatrix}. \tag{4.7}$$

The coordinates of the vertices $x_1,\ldots,x_4$ of P equal the columns of $\mathbf{X}$ (see 3.2). This projection P is the bipyramid D_3' based on a regular hexagon, and it does have the combinatorial type of Fig. 4.1 (respecting also the given labels). Thus $G(\Phi)\subset\chi(\Phi)$, and P has the largest volume among projections of X^4 with the same type as Fig. 4.1.

Repeating this process for Fig. 4.2, it turns out that the comass is again $2/\sqrt{3}$. However, the projection attaining this volume is not simplicial. Thus $2/\sqrt{3}$ is an upper bound on the volume of projections of this type.

Here we calculate the comass for Fig. 5.1, which is a projection of X^5 into R^3. Its form is

$$\begin{aligned}\Phi &= \tfrac{2}{3}(e_{123}+e_{134}+e_{145}+e_{125}) = (\tfrac{2}{3}e_1)\wedge\Psi,\\ &\text{where } \Psi = e_{23}+e_{34}+e_{45}+e_{25}.\end{aligned} \tag{4.8}$$

Now, $\Psi\in\Lambda_2 L$ where $L=\operatorname{lin}\{e_2,\ldots,e_5\}$ is isomorphic to R^4. Hence, $\|\Psi\|$ on L can be calculated according to [10, pp. 8–9]. The result is that $\|\Psi\|=\sqrt{2}$ and

$$\begin{aligned}G(\Psi) &= \frac{1}{\sqrt{2}}(\tfrac{1}{2}+a_1)e_{23}+(\tfrac{1}{2}-a_1)e_{45}+a_2e_{24}+a_2e_{35}\\ &\quad +(\tfrac{1}{2}+a_3)e_{25}+(\tfrac{1}{2}-a_3)e_{34},\end{aligned} \tag{4.9}$$

where $\Sigma a_i^2=\tfrac{1}{2}$. Therefore $G(\Phi)=e_1\wedge G(\Psi)$ and $\|\Phi\|=2\sqrt{2}/3$. A 3×5 matrix $\mathbf{X}$ whose rows span a 3-plane L in $G(\Phi)$ is determined by a similar method as in (4.4) through (4.7). Assuming $a_2\neq 0$, one such matrix is

$$\mathbf{X}=\frac{1}{\sqrt{2}\,a_2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & (\tfrac{1}{2}-a_3) & 0 & -(\tfrac{1}{2}-a_1) \\ 0 & 0 & (\tfrac{1}{2}+a_1) & a_2 & (\tfrac{1}{2}+a_3) \end{bmatrix}.$$

When $a_1 = a_3 = 0$ and $a_2 = 1/\sqrt{2}$, $\mathbf{X}$ is an orthogonal matrix and the columns of $\mathbf{X}$ give the vertices of the projection into L, which in this case is D_4'. Varying the numbers a_1, a_2 and a_3 gives a continuous family of projections of X^5 which all have the largest volume within this combinatorial type.

Calculation of the comass of forms in $\Lambda_3 R^6$ in general relies on a computer gradient method described below. Given $\Phi \in \Lambda_3 R^6$, we seek to find an orthonormal basis $e_1', \ldots, e_6'$ of R^6 and a $\Phi' \in \Lambda_3 R^6$ of the form

$$
\begin{aligned}
\Phi' = e_{123}' &+ a_{11} e_{156}' + a_{12} e_{416}' + a_{13} e_{451}' + a_{21} e_{256}' + a_{22} e_{426} \\
&+ a_{23} e_{452}' + a_{31} e_{356}' + a_{32} e_{436}' + a_{33} e_{453}' + \mu e_{456}',
\end{aligned} \tag{4.10}
$$

such that $|\Phi - \alpha\Phi'|$ is small, $\alpha \in R$. A method for finding this basis is given in [10, Theorem 4.1]. The main step is to find a critical point of the function $\langle \xi, \Phi \rangle$, $\xi \in G(3, 6)$, which can be done on the computer using a gradient technique. Then we apply a simple test due to Dadock and Harvey (see [4]). They show that Φ' has comass 1 (and hence $e_{123}' \in G(\Phi')$) if and only if

$$1 \geqq |\mathbf{A}|^2 + 2 \det \mathbf{A} + \mu^2 \tag{4.11}$$

and

$$1 \geqq |\det \mathbf{A}|, \tag{4.12}$$

where $\mathbf{A} = (a_{ij})$ and $|\mathbf{A}|^2 = \Sigma\, a_{ij}^2$.

Now, definition (1.10) and the fact that $\|\cdot\|$ is a norm on $\Lambda_3 R^6$ imply

$$\big|\, \|\Phi\| - \|\alpha\Phi'\| \,\big| \leqq \|\Phi - \alpha\Phi'\| \leqq |\Phi - \alpha\Phi'|. \tag{4.13}$$

Hence if conditions (4.11) and (4.12) hold, then the difference between $\|\Phi\|$ and α is bounded by $|\Phi - \alpha\Phi'|$. Tables 3 and 6 were calculated using this procedure. In every case conditions (4.11) and (4.12) were met and the error $|\Phi - \alpha\Phi'|$ was at most 10^{-13}. Among projections of X^6 into R^3, type 6.21 has the largest comass, and it turns out that D_5' is a critical point which represents a local, but not global maximum of $\langle \Phi, \cdot \rangle$. More detailed calculations imply that most forms in Table 6 have no critical points of the corresponding combinatorial type.

We conclude with the calculation of $\|\Phi\|$, where Φ is the projection form of the regular icosahedron P. The symmetry group of P implies that $P = \Pi(X^6 : L)$ for some 3-plane L in R^6 (see [3, 13.91]). Moreover, the 3-vector ξ corresponding to L is a critical point of $\langle \Phi, \cdot \rangle$ by [8, Theorem 8]. The projection $\Pi(X^6 : L^\perp)$ into the 3-plane $L^\perp$ orthogonal to L is also a regular icosahedron, and $\xi^\perp$ is another critical point of $\langle \Phi, \cdot \rangle$.

If $e'_1, \ldots, e'_6$ is an orthonormal basis of R^n with $\xi = e'_{123}$, then the remarks above imply $\langle \Phi, e'_{ijk} \rangle = 0$ whenever $|\{i, j, k\} \cap \{1, 2, 3\}| = 2$ or $|\{i, j, k\} \cap \{4, 5, 6\}| = 2$ (see [8, (4.13)]). Hence Φ can be written as

$$\Phi = \alpha(e'_{123} + (\tau - 1)e'_{456}), \tag{4.4}$$

where $\alpha = (2\sqrt{2}/3)\sin(2\pi/5)$ is the volume of P, and $\tau = (1 + \sqrt{5})/2$ is the golden ratio. The matrix $\mathbf{A}$ in (4.11) equals the zero matrix in this case, so (4.11) and (4.12) are satisfied. Therefore $\|\Phi\| = \alpha$ and P is the largest projection of its type.

5. Upper bounds

The results in the previous sections suggest that the maximal projections stabilize after some n. The conjecture for 3-dimensional projections is that $\operatorname{Max} \Pi^3(T^n) = D_4$, $n \geqq 5$, and $\operatorname{Max} \Pi^3(X^n) = X^3$, $n \geqq 4$. In this section we shall establish upper bounds which support these conjectures, and in the case of X^n show that the truth of the conjecture for $n \leqq 42$ would imply that it holds for all n.

First, we shall need a condition for $V(T^n : L)$, $L \in G(k, n)$, to have a critical point at L. Let $P = \Pi(T^n : L) = \operatorname{conv}\{x_0, \ldots, x_n\}$. From [8, Theorem 8] we obtain

$$|x_i|^2 = \alpha |x_i| V(P_i), \tag{5.1}$$

where $\alpha \in R$ and P_i is the projection of a part of P into a hyperplane normal to x_i. Substituting (5.1) in the formula

$$V(P) = \frac{1}{k^2} \Sigma |x_i| V(P_i)$$

gives $\alpha = 1/kV(P)$. If $c_i = |x_i|$, $V = V(P)$ and $V_i = V(P_i)$, (5.1) then becomes

$$c_i = V_i / kV. \tag{5.2}$$

Let B^k be the unit ball centered at the origin in R^k, and let $\omega_k = V(B^k)$.

THEOREM 1. *If* $P = \Pi(T^n : L)$, $L \in G(k, n)$, *with* k *and* $n \geqq 3$, *then*

$$V(P) \leqq f(k) \tag{5.3}$$

where

$$f(k) = \left(\frac{\omega_3\omega_4\cdots\omega_k}{3^3 4^4\cdots k^k}\right)^{1/(k+1)}. \tag{5.4}$$

Proof. This inequality will be proved by induction on k. Let $k = 3$, and suppose x_1 has the greatest norm c_1 among the vertices of P. Then $P \subset c_1B^3$ and the volume V of P satisfies

$$V \leqq \omega_3 c_1^3. \tag{5.5}$$

Since P_1 is a planar projection of T^n, formula (2.4) implies $V_1 \leqq 1$. Substituting into (5.2) gives

$$c_1 \leqq 1/3V. \tag{5.6}$$

Combining (5.5) and (5.6), we obtain

$$V \leqq \omega_3(1/3V)^3$$

or

$$V \leqq (\omega_3/3^3)^{1/4}. \tag{5.7}$$

With (5.3) proved for $k = 3$, we assume it holds for $k \geqq 3$ and show that it holds for $k + 1$. Letting $P = \Pi(T^n : L)$, $L \in G(k + 1, n)$, and using the same notation as above, $P \subset c_1B^{k+1}$ and

$$V \leqq \omega_{k+1} c_1^{k+1}. \tag{5.8}$$

In this case, P_1 is a projection of T^n into a k-plane, so by induction

$$V \leqq f(k). \tag{5.9}$$

Combining (5.2) and (5.9), and substituting in (5.8) gives the required result. ■

A better bound is obtained when we assume the projection has the greatest possible number of vertices.

Theorem 2. *Let $P = \Pi(T^n : L)$, $L \in G(k, n)$, with k and $n \geqq 3$, and suppose P has $n + 1$ vertices. Then*

$$V(P) \leqq \left[\frac{\omega_k}{2k^k}\left(f(k-1) + \frac{2\omega_{k-1}}{\omega_k}\sqrt{\frac{k}{n+1}}\, f(k)\right)^k\right]^{1/(k+1)}, \tag{5.10}$$

where $f(k)$ is 1 when $k = 2$ and is given by (5.4) when $k \geqq 3$.

PROOF. Our notation will be the same as in the previous theorem. Formula (2.1) implies that the average norm of a vertex of P is $\varepsilon = \sqrt{k/n+1}$. Hence, there must be a supporting hyperplane of P whose distance from the origin is less than ε. This means that P is contained in the union of a hemisphere of radius c_1 and a cylinder of height ε. Therefore

$$V \leqq \frac{\omega_k}{2} c_1^k + \omega_{k-1} c_1^{k-1} \varepsilon \leqq \frac{\omega_k}{2}\left(c_1 + \frac{2\omega_{k-1}}{k\omega_k}\varepsilon\right)^k. \tag{5.11}$$

From Theorem 1,

$$c_1 \leqq \frac{f(k-1)}{kV} \quad \text{and} \quad \varepsilon = \frac{\varepsilon V}{V} \leqq \frac{\varepsilon f(k)}{V}.$$

Substituting these two inequalities in (5.11) gives (5.10). ■

When $k = 3$, the bounds of the two theorems are

$$V \leqq \frac{\sqrt{2}}{3}\pi^{1/4} \tag{A}$$

and

$$V \leqq \frac{\sqrt{2}}{3}\left[\frac{\pi}{2}\left(1 + \sqrt{\frac{3\sqrt{\pi}}{2(n+1)}}\right)^3\right]^{1/4} \tag{B}$$

respectively. Note that $\sqrt{2}/3 = V(D_4)$ is the conjectured maximum for V. As n approaches infinity, the bound in (B) decreases towards $(\sqrt{2}/3)(\pi/2)^{1/4}$, and is smaller than the one in (A) when $n \geqq 39$.

A final bound in 3 dimensions can be obtained as in Theorem 1, except that we substitute a formula of Fejes Tóth ([6, p. 264]) in place of (5.5).

THEOREM 3. *If $P = \Pi(T^n : L)$, $L \in G(3, n)$ has $n+1$ vertices, then*

$$V(P) \leqq \left(\frac{g(n+1)}{27}\right)^{1/4}, \tag{C}$$

where

$$g(n) = \frac{(n-2)}{6}\cot\theta_n(3 - \cot^2\theta_n) \quad \textit{and} \quad \theta_n = \frac{n}{n-2}\,\frac{\pi}{6}.$$

This bound is good for small n, and in fact equality holds when $n = 5$. As n increases, it approaches the right-hand side of (A). When $n \geqq 55$, the best bound of all three is the one in (B).

The critical condition which replaces (5.2) for projections of X^n is

$$V = 2V_i/kc_i. \tag{5.12}$$

Repeating Theorem 1 with this modification yields

THEOREM 4. *If $P = \Pi(X^n : L)$, $L \in G(k, n)$, with k and $n \geqq 3$, then*

$$V(P) \leqq 2^{k/2} f(k). \tag{5.13}$$

A much stronger result holds in place of Theorem 2.

THEOREM 5. *Let $P = \Pi(X^n : L)$, $L \in G(k, n)$, with k and $n \geqq 3$, and suppose P has $2n$ vertices. Then*

$$V(P) \leqq 2\omega_{k-1}\sqrt{k/n} \tag{5.14}$$

PROOF. For convenience, we shall assume that the vertex x_1 of P has norm $c_1 = 1$, which is the greatest possible. By the same argument as in Theorem 2, there must be a supporting hyperplane of P within $\varepsilon = \sqrt{k/n}$ of the origin. But P is centrally symmetric, and so is contained in a cylinder whose base B is a unit $(k-1)$-ball and whose side S has height 2ε. We may then write

$$x_1 = \frac{u + \alpha v}{\sqrt{1+\alpha^2}}, \qquad \alpha \leqq \varepsilon, \tag{5.15}$$

where u and v are unit vectors with u parallel to B and v parallel to S.

Now we determine the volume of the projection of this cylinder into the hyperplane H_1 orthogonal to x_1. The projections of the base and side satisfy

$$V(B : H_1) = (\omega_{k-1} v) \cdot x_1 = \frac{\alpha\omega_{k-1}}{\sqrt{1+\alpha^2}},$$

and

$$V(S : H_1) = ((k-1)\omega_{k-1}\varepsilon u) \cdot x_1 = \frac{(k-1)\omega_{k-1}\varepsilon}{\sqrt{1+\alpha^2}}.$$

Since P_1 is contained in the cylinder,

$$\begin{aligned} V_1 \leqq V(B : H_1) + V(S : H_1) &= \frac{\omega_{k-1}\alpha + (k-1)\omega_{k-1}\varepsilon}{\sqrt{1+\alpha^2}} \\ &\leqq k\omega_{k-1}\varepsilon. \end{aligned} \tag{5.16}$$

The theorem is completed by substituting (5.16) in (5.12). ∎

The 3 dimensional bounds for projections of X^n corresponding to (A) through (C) are

(A′) $$V \leqq \tfrac{4}{3}\pi^{1/4},$$

(B′) $$V \leqq 2\pi\sqrt{3/n},$$

and

(C′) $$V \leqq \left(\frac{8g(2n)}{27}\right)^{1/4}.$$

Equality holds in (C′) when $n = 3$, and the right-hand side of (A′) is its limit as n approaches infinity. However, the bound in (B′) goes to zero as n gets large. Using somewhat finer estimates for ε in Theorem 5, it can be shown that $V(X^n : R^3) < 4/3 = V(X^3)$ when $n > 42$. Hence, a maximal projection of X^n into R^3 can have at most 84 vertices.

6. Tables

The first two tables list the results on maximal projections from Sections 2, 3 and 4 (see also [9]). The other tables give upper bounds on the volume of various combinatorial types of projections into R^3. The asterisk * indicates that the numbers in the corresponding column are given to the nearest thousandth. A $\surd$ means that the bound is attained by a figure of this type. Figures 3.1 through 3.5 are Schlegel diagrams, and the remaining figures are half the Schlegel diagrams of centrally symmetric polytopes. The results for projections of C^n can be found in [2] and [7].

TABLE 1
The largest projections into R^2

	Projection	Volume
T^n, $n \geqq 3$	C^2	1
X^n, $n \geqq 2$	X^2	2
C^n, $n \geqq 2$	$\sqrt{2/n}\,R_n$	$\cot(\pi/2n)$

TABLE 2
The largest projections into R^3

	Projection	Volume
T^4	D_3	$1/\sqrt{6}$
T^5, T^6	D_4	$\sqrt{2}/3$
X^4, X^5, X^6	X^3	4/3
C^4	rhombic dodecahedron	2
C^5	stretched rhombic icosahedron [3, p. 257]	$\cot(\pi/10)$
C^6	triacontahedron	$\sqrt{2}\cot(\pi/10)$

TABLE 3
Simplicial 3-polytopes with 7 vertices

Figure	Comass*
3.1	0.382 $\surd$
3.2	0.399 $\surd$
3.3	0.408
3.4	0.441 $\surd$
3.5	0.448 $\surd$

TABLE 4
Centrally symmetric, simplicial 3-polytopes with 8 vertices

Figure	Comass
4.1	$2\sqrt{3}/3$ $\surd$
4.2	$2\sqrt{3}/3$

TABLE 5
Centrally symmetric, simplicial 3-polytopes with 10 vertices

Figure	Comass*
5.1	0.943 $\surd$
5.2	0.980
5.3	1.007 $\surd$
5.4	1.032
5.5	1.112

TABLE 6
Centrally symmetric, simplicial 3-polytopes with 12 vertices

Figure	Comass*
6.1	0.880 √
6.2	0.880 √
6.3	0.890
6.4	0.903
6.5	0.911
6.6	0.916
6.7	0.930
6.8	0.943
6.9	0.943
6.10	0.943
6.11	0.961
6.12	0.966
6.13	0.990
6.14	1.020
6.15	1.034
6.16	1.060
6.17	1.096
6.18	1.118
6.19	1.133
6.20	1.177
6.21	1.268

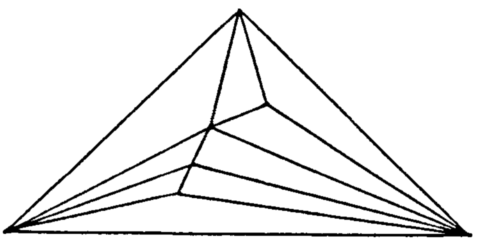

Fig. 3.1

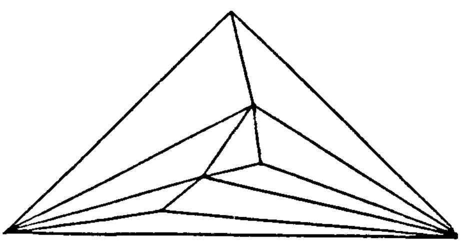

Fig. 3.2

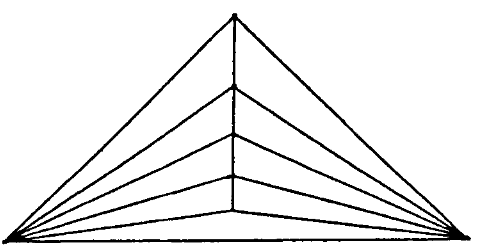

Fig. 3.3

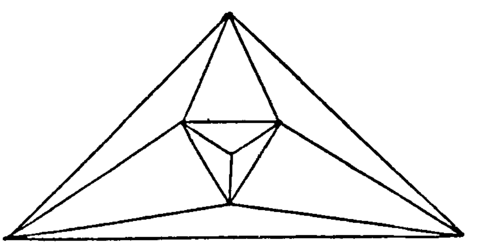

Fig. 3.4

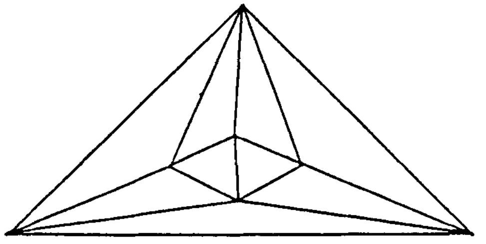

Fig. 3.5

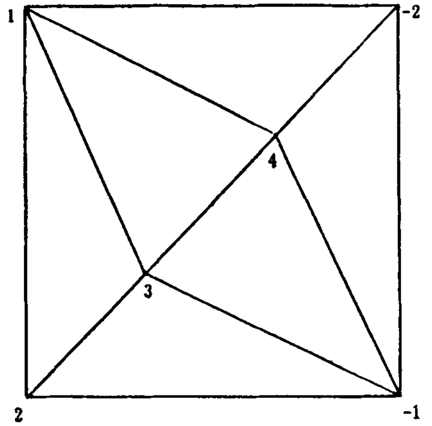

Fig. 4.1

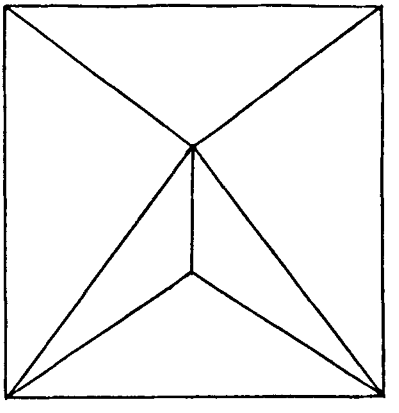

Fig. 4.2

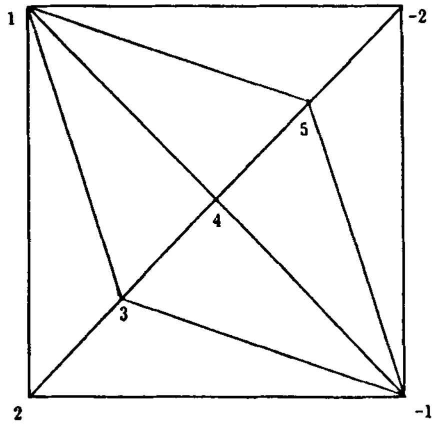

Fig. 5.1

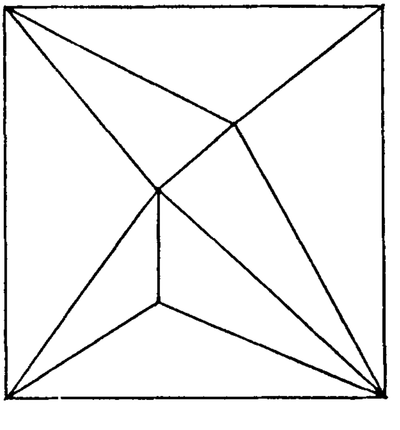

Fig. 5.2

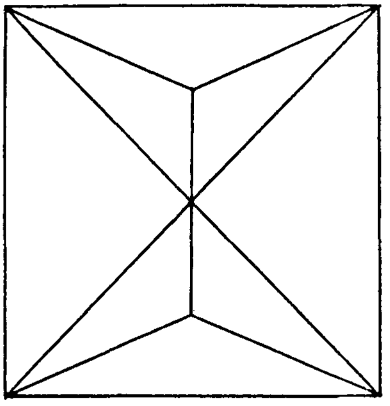

Fig. 5.3

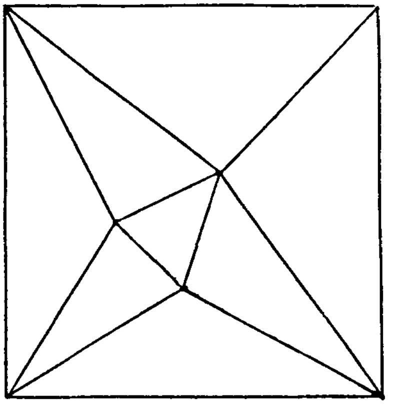

Fig. 5.4

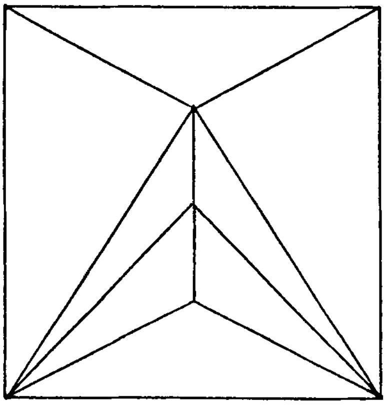

Fig. 5.5

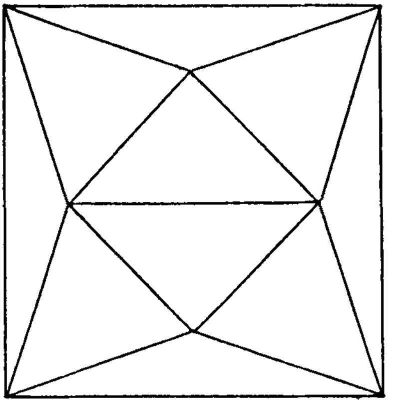

Fig. 6.1

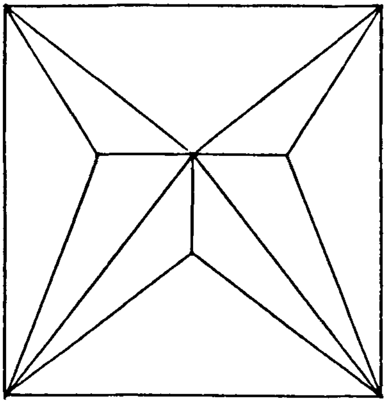

Fig. 6.2

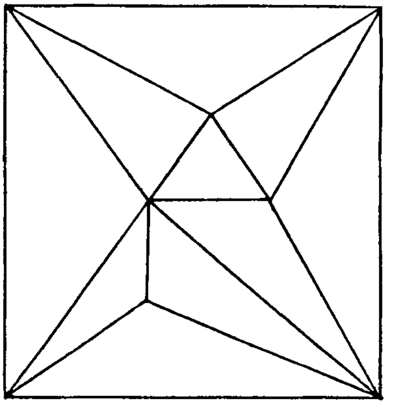

Fig. 6.3

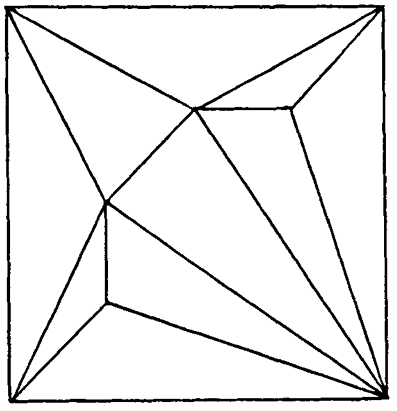

Fig. 6.4

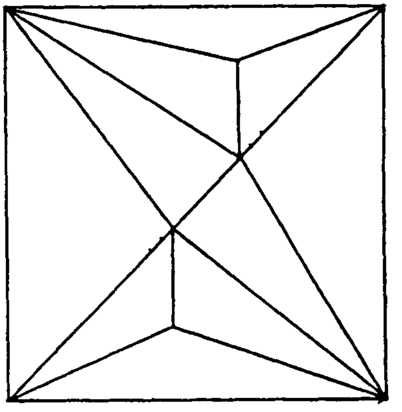

Fig. 6.5

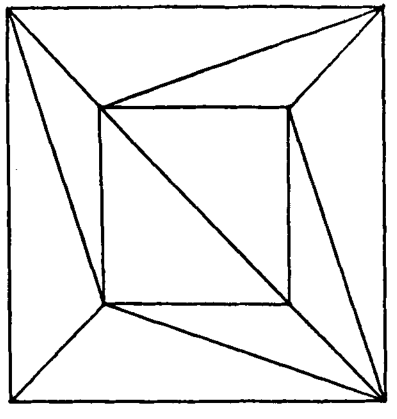

Fig. 6.6

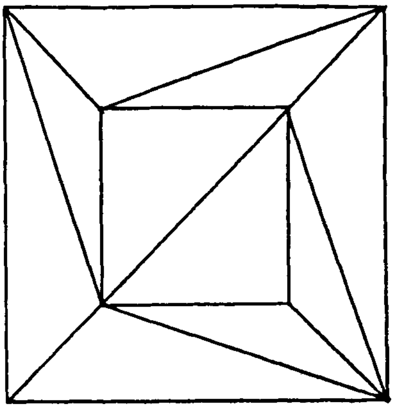

Fig. 6.7

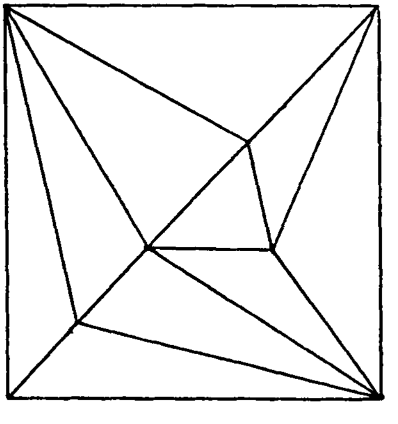

Fig. 6.8

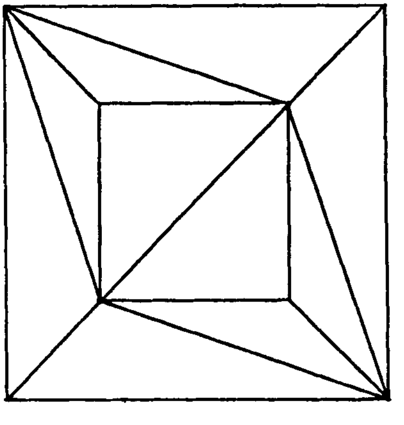

Fig. 6.9

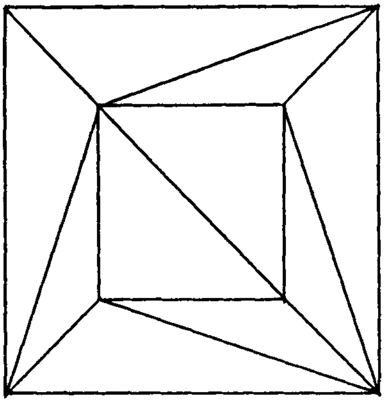

Fig. 6.10

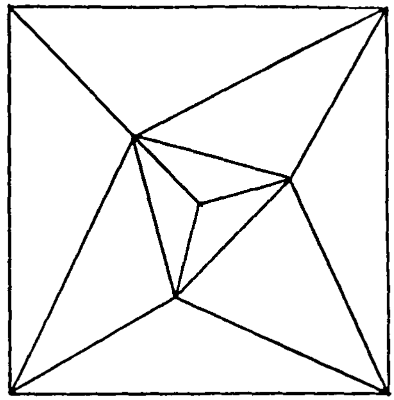

Fig. 6.11

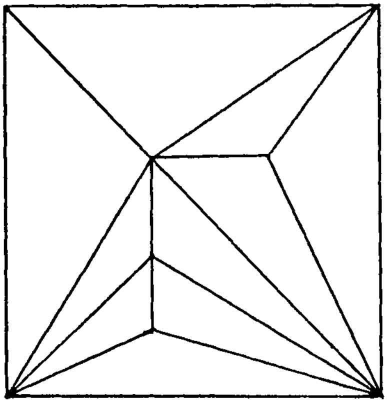

Fig. 6.12

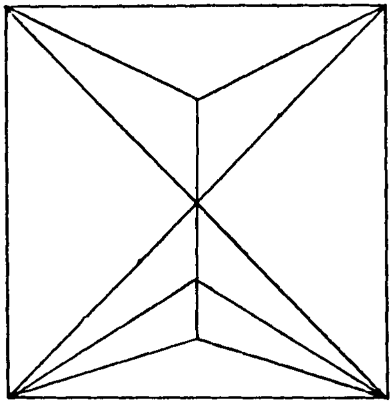

Fig. 6.13

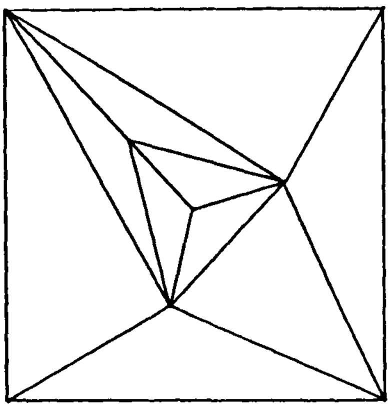

Fig. 6.14

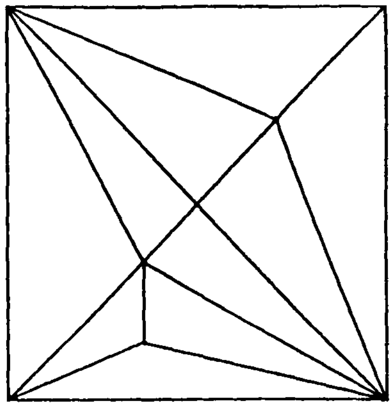

Fig. 6.15

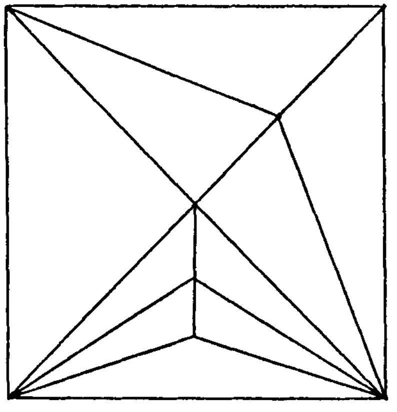

Fig. 6.16

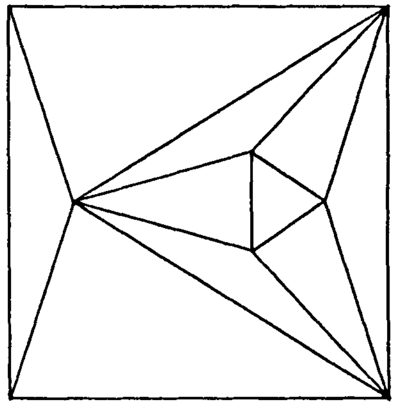

Fig. 6.17

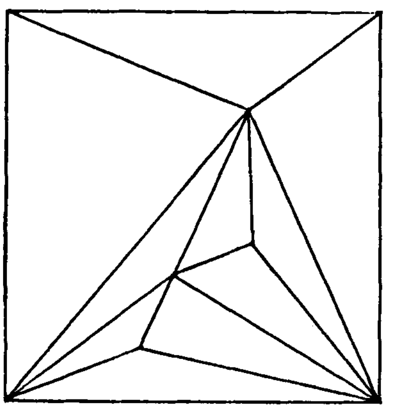

Fig. 6.18

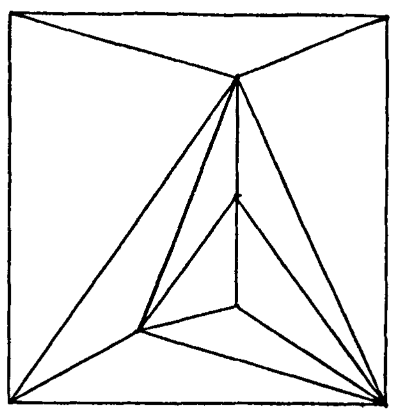

Fig. 6.19

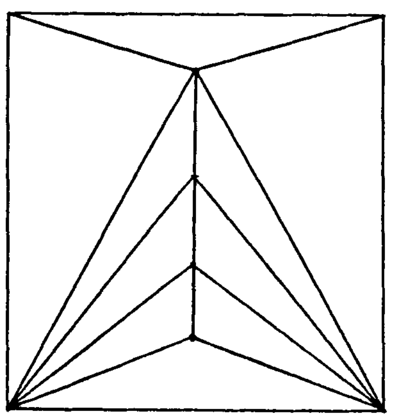

Fig. 6.20

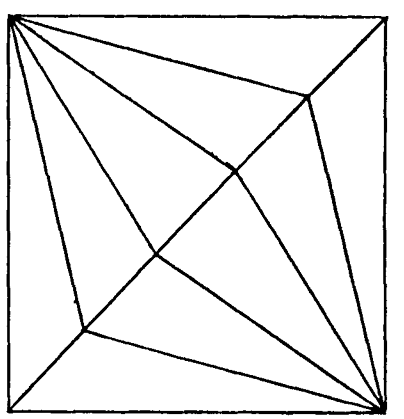

Fig. 6.21

References

1. J. D. Berman and K. Hanes, *Volumes of polyhedra inscribed in the unit sphere in E^3*, Math. Ann. **188** (1970), 78–84.

2. G. D. Chakerian and P. Filliman, *The measures of the projections of a cube*, Stud. Sci. Math. Hung. **21** (1986), 103–110.

3. H. S. M. Coxeter, *Regular Polytopes*, Dover, New York, 1973.

4. J. Dadock and R. Harvey, *Calibrations on R^6*, Duke Math. J. **50** (1983), 1231–1243.

5. H. Federer, *Geometric Measure Theory*, Springer-Verlag, Berlin, 1969.

6. L. Fejes Tóth, *Regular Figures*, MacMillan, New York, 1964.

7. P. Filliman, *Extremum problems for zonotopes*, Geom. Dedic. **27** (1988), 251–262.

8. P. Filliman, *Projections of Polytopes*, Discrete Comp. Geom., to appear.

9. H. Martini and B. Weissbach, *Zur besten Beleuchtung Konvexer Poly eder*, Beitr. Alg. Geom. **17** (1984), 151–168.

10. F. Morgan, *The exterior algebra $\Lambda^k R^n$ and area minimization*, Linear Algebra & Appl. **66** (1985), 1–28.